§ Motivation

AGT conjecture (= 'Th' [Braverman-Finkelberg-N])

$\mathcal{U}_G^d$: Uhlenbeck partial cptification of framed moduli of instantons on $\mathbb{R}^4$ with $\begin{cases} \text{gauge grp} & G \\ \text{instanton \#} & d \end{cases}$

Assume G: ADE

$\Rightarrow$ Equivariant intersect. cohom. $\bigoplus_d IH^*_{\mathbb{T}}(\mathcal{U}_G^d)$ has a structure of a representation of $W_k(\mathfrak{g})$: W-algebra for Lie G

where $\begin{cases} k + h^\vee = -\varepsilon_2/\varepsilon_1 & \text{with} \quad \varepsilon_1, \varepsilon_2 : \text{equiv. parameter} \\ \text{highest weight} = \vec{a} \end{cases}$

$\mathbb{T} = T \times T^2 \qquad H^*_{\mathbb{T}}(pt) = \mathbb{Q}[\varepsilon_1, \varepsilon_2, \vec{a}] =: A_T \qquad a = (a_1, \cdots, a_\ell)$

(a polynomial ring!)

Observations

1) $IH^*_{\mathbb{T}}(\mathcal{U}_G^d)$ is a free module over A_T : polynomial ring.

2) $IH^{*+2dh^\vee}_{\mathbb{T}}(\mathcal{U}_G^d)$ has the (perverse) cohomological grading.

But 1), 2) don't! $\longrightarrow$ Even a statement must be clarified!

$\longrightarrow$ motivation for today

$\Longrightarrow$ Representations of the W-algebra must be defined over $A_T = \mathbb{Q}[\varepsilon_1, \varepsilon_2, \vec{a}]$

Biproduct: One can make specialization

at $\varepsilon_1 = 0 \longrightarrow$ commutative, but still complicated

or $\varepsilon_1 = \varepsilon_2 = 0 \longrightarrow$ simpler!

future application : Understand irr. modules at arbitrary central charge / highest wt

Today I will focus on the A-form of the W-algebra, not representations.

But mention the result and its geometric background briefly:

We construct four A_T-modules

$$\begin{pmatrix}\text{universal}\\ \text{Verma}\end{pmatrix} \subset \begin{pmatrix}\text{universal}\\ \text{Wakimoto}\end{pmatrix} \subset \begin{pmatrix}\text{universal}\\ \text{dual Wak}\end{pmatrix} \subset \begin{pmatrix}\text{universal}\\ \text{dual Verma}\end{pmatrix}$$

— They are all the same over $\otimes_{A_T} \mathrm{Frac}(A_T)$, i.e.,

generic central charge / highest weight

— At specialization at nongeneric $\varepsilon_1, \varepsilon_2, \vec{a}$, they are different, and are Verma / Wakimoto / dual Wak / dual Verma.

They correspond

$$\oplus IH^*_{T,c}(\mathcal{U}^d_G) \subset \oplus H^*_{T,c}\begin{pmatrix}\text{hyperbolic}\\ \text{restriction}\end{pmatrix} \subset \oplus H^*_{T}(\text{hyp. rest}) \subset \oplus IH^*(\mathcal{U}^d_G)$$

compact support

<u>Rem</u>. A_T-forms are important, when we prove the Whittaker property.

§ integral forms of Heis. & Virasoro algebras

$\mathbb{A} = \mathbb{Q}[\varepsilon_1, \varepsilon_2]$, $\mathbb{F} = \mathbb{Q}(\varepsilon_1, \varepsilon_2)$

$(\mathbb{A}_T = \mathbb{Q}[\varepsilon_1, \varepsilon_2, \vec{a}]$ $\vec{a} = (a_1, \cdots, a_\ell)$ above$)$

Virasoro alg. rel.

$$[L_m, L_n] = (m-n) L_{m+n} + \left(1 + \frac{6(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}\right) \frac{m^3-m}{12} \delta_{m,-n}$$

Let us introduce a modified generator $\tilde{L}_m := \varepsilon_1\varepsilon_2 L_m$ to remove $\varepsilon_1\varepsilon_2$ from the denom.

$$\therefore [\tilde{L}_m, \tilde{L}_n] = (m-n)\varepsilon_1\varepsilon_2 \tilde{L}_{m+n} + \{(\varepsilon_1\varepsilon_2)^2 + 6(\varepsilon_1+\varepsilon_2)^2 \varepsilon_1\varepsilon_2\} \frac{m^3-m}{12} \delta_{m,-n}$$

Heis. alg. rel. (coupled with $[\mathbb{C}^2]$)

$$[P_m, P_n] = \frac{1}{\underbrace{\varepsilon_1\varepsilon_2}_{}} m \delta_{m,-n}$$

$$\left(\frac{1}{\varepsilon_1\varepsilon_2} = \int_{\mathbb{C}^2} 1 \right)$$

Let $\tilde{P}_m = \varepsilon_1\varepsilon_2 P_m$.

$$\therefore [\tilde{P}_m, \tilde{P}_n] = \varepsilon_1\varepsilon_2 m \delta_{m,-n}$$

Also $\tilde{L}_n = -\sum :\tilde{P}_m \tilde{P}_{n-m}: - (n+1)(\varepsilon_1+\varepsilon_2)\tilde{P}_n$

These $\mathbb{A}$-forms are natural for specialization $\varepsilon_1, \varepsilon_2 = 0$

- $\varepsilon_1 = 0$ $\Rightarrow$ Heis. Vir. become commutative

  Get Poisson str. ~ related to integrable hierarchy

  $\tilde{L}_n$ still contain $-(n+1)\varepsilon_2 \tilde{P}_n$

- $\varepsilon_1 = \varepsilon_2 = 0$ $\Rightarrow$ $\tilde{L}_n := -\sum \tilde{P}_m \tilde{P}_{n-m}$ 1st inv. pol

Introduce 'coh deg' on $\mathrm{Heis}_{\mathbb{A}}$ & $\mathrm{Vir}_{\mathbb{A}}$ by

$$\mathrm{cohdeg}\, \varepsilon_1 = \mathrm{cohdeg}\, \varepsilon_2 = 1$$
$$\mathrm{cohdeg}\, \widetilde{P}_m = 1 \quad , \quad \mathrm{cohdeg}\, \widetilde{L}_n = 2$$

$\Longrightarrow$ relations are homogeneous

$\mathrm{Vir}_{\mathbb{A}} \hookrightarrow \mathrm{Heis}_{\mathbb{A}}$ are graded alg. and

$\hookrightarrow$ is grade preserving

§ Integral forms of W-algebras

$W_k(\mathfrak{g})$: W-algebra of level k

We consider k as a variable. ↖ vertex alg. $/\mathbb{Q}(k)$

Main Result 1

An explicit construction (as a BRST cohom) of

$W_A(\mathfrak{g})$: vertex algebra over $A = \mathbb{Q}[\varepsilon_1, \varepsilon_2]$

s.t. $\quad W_A(\mathfrak{g}) \otimes_A \mathbb{F} \cong W_k(\mathfrak{g}) \otimes_{\mathbb{Q}(k)} \mathbb{F}$

<u>properties</u>

0) $W_A(\mathfrak{g})$ is free over A

1) $\exists$ PBW A-base from $W^{(a)}(z)$ $\quad a = 1, \cdots, \ell$

2) $\exists$ cohomological grading $\qquad$ cohdeg $\varepsilon_1, \varepsilon_2 = 1$

cohdeg $W^{(a)}(z) = d_a + 1$

$d_1, \cdots, d_\ell$: exponents of $\mathfrak{g}$

3) $W_A(\mathfrak{g})\big|_{\varepsilon_1 = 0} \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}[$moduli of $(-\varepsilon_2)$-G-opers on $D = \operatorname{Spec} \mathbb{C}[[t]]]$

$\nabla = (-\varepsilon_2) d + A$

3)' $W_A(\mathfrak{g})\big|_{\varepsilon_1 = \varepsilon_2 = 0} \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}[$moduli of 0-G-opers on $D]$

$$\begin{array}{ccl} \text{moduli of } 0\text{-opers} & \subset & \mathcal{H}igg(D) : \text{moduli of } G\text{-Higgs bdles on } D \\ & \overset{\cong}{\searrow} & \downarrow \text{ Hitchin Hamiltonian} \\ & & \mathcal{H}it(D) : \text{Hitchin base} \\ & & \| \\ & & \bigoplus_{a=1}^{\ell} H^0(D, \Omega_D^{\otimes(d_a+1)}) \end{array}$$

$$\text{RHS} \cong \mathbb{C}[\mathcal{H}igg(D)] \cong \mathbb{C}[\mathfrak{g}[[z]]]^{G[[z]]}$$

Remark (grading vs filtration)

Let $\mathbb{B} = \mathbb{Q}[\varepsilon_1] \rightsquigarrow \frac{1}{k+h^\vee}$ if $\varepsilon_2 = -1$

$$\Longrightarrow \quad W_{\mathbb{A}}(\mathfrak{g}) \otimes_{\mathbb{A}} \mathbb{B} \quad (\mathbb{A} \nearrow \varepsilon_2 = -1)$$ is a filtered vertex alg.

induced from the grading on $W_{\mathbb{A}}(\mathfrak{g})$

whose ass. graded $= W_{\mathbb{A}}(\mathfrak{g}) \otimes_{\mathbb{A}} \mathbb{Q}[\varepsilon_1]$ $(\mathbb{A} \nearrow \varepsilon_2 = 0)$

From + filtr., the $\mathbb{A}$-form $W_{\mathbb{A}}(\mathfrak{g})$ is recovered by the Rees algebra construction

(sketch of the construction)

$$W_k(\mathfrak{g}) = H^0(C_k(\mathfrak{g}), d = d_{st} + \chi)$$

$$C_k(\mathfrak{g}) = C_k(\mathfrak{g})_0 \otimes C_k(\mathfrak{g})'$$

construct an $\mathbb{A}$-form $C_{\mathbb{A}}(\mathfrak{g})_0$ there

Modify $\tilde{\chi} = \chi/\varepsilon_1$ to preserve the $\mathbb{A}$-form.

§ Screening operators

spectral sequence associated with double cpx :

$$H^\bullet(\underline{H^\bullet(C_k(\mathfrak{g})_0, d_{st})}, \chi) \Rightarrow H^\bullet(C_k(\mathfrak{g})_0, d_{st}+\chi)$$

$$\underline{H^\bullet(C_k(\mathfrak{g})_0, d_{st})} = \widetilde{H}^\bullet_k(\mathfrak{g})$$

k: generic $\Rightarrow$ degenerate at E_2

$$W_k(\mathfrak{g}) = \mathrm{Ker}(\chi : \widetilde{H}^0_k(\mathfrak{g}) \to \widetilde{H}^1_k(\mathfrak{g}))$$

- Heisenberg alg
- its module
- χ : screening operator

<u>A-form</u> ?

a slight problem $H^\bullet(C_A(\mathfrak{g})_0, d_{st})$ is <u>not</u> free over A.

Let $B := \mathbb{Q}[\varepsilon_1]$ as before

$$\underset{\nearrow\, \varepsilon_2 = -1}{C_A(\mathfrak{g})_0 \otimes_A B} \quad \rightsquigarrow_{\text{coh. w.r.t. } d_{st}} \quad \widetilde{H}^\bullet_B(\mathfrak{g})$$

This has a filtration, induced from the grading on $C^\bullet_A(\mathfrak{g})_0$.

<u>Prop</u> 1) $\widetilde{H}^0_B(\mathfrak{g})$: B-form of the Heisenberg alg

2) $\widetilde{H}^{1,0}_B(\mathfrak{g})$: B-form of its module

& $\chi : \widetilde{H}^0_B(\mathfrak{g}) \to \widetilde{H}^{1,0}_B(\mathfrak{g})$

B-form of the screening operator

$$\begin{tikzcd} C^{0,0} \arrow[r, "\chi"] & C^{1,0} \arrow[r] & C^{2,0} \\ & C^{1,-1} \arrow[u] \arrow[r] & C^{2,-1} \arrow[u] \\ & & C^{2,-2} \arrow[u] \end{tikzcd}$$

? $\widetilde{H}^{i+1,-i}_B(\mathfrak{g}) = 0$ for $i > 0$

torsion over B

$$W_{\mathbb{A}}(\mathfrak{g}) \hookrightarrow \mathrm{Rees}(\mathrm{Ker}(\mathfrak{X} : \tilde{H}^{0}_{\mathbb{B}}(\mathfrak{g}) \to \tilde{H}^{1,0}_{\mathbb{B}}(\mathfrak{g})))$$

$?$ $\subset$ "$=$"

specializations

1) $H^{0}_{\mathbb{B}}(\mathfrak{g}) \otimes_{\mathbb{B}} \mathbb{C}_{\varepsilon_1 = 0} = \mathbb{C}[$moduli sp. of generic Miura opers on $D]$

2) $\tilde{H}^{0}_{\mathbb{B}}(\mathfrak{g}) \otimes_{\mathbb{B}} \mathbb{C}$ with the filtration.

The Rees alg. construction

$\Longrightarrow$ moduli space of generic Miura $(-\varepsilon_2)$-opers on D

associated gr., i.e. $\varepsilon_2 = 0$

moduli space of generic Miura 0-opers

At $\varepsilon_1 = \varepsilon_2 = 0$

$$\begin{array}{ccc} W_{\mathbb{A}}(\mathfrak{g}) & \longrightarrow & \mathrm{Heis}_{\mathbb{A}}(\mathfrak{g}) \\ \| & & \| \\ \mathbb{C}[\mathfrak{g}[[z]]]^{G[[z]]} & & \mathbb{C}[\mathfrak{h}[[z]]] \end{array}$$ is given by

generalization of $\quad L_n = \sum P_m P_{n-m}$

$$\sum_{n<0} W^{(a)}_n z^n = E^{(a)}\Big(\sum_{n<0} P_n z^n\Big)$$

a^{th} inv. poly